The thermal phase transition in a QCD-like holographic model

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We investigate the high temperature phase of a dilaton flow deformation of the AdS/CFT Correspondence. We argue that these geometries should be interpreted as the $\mathcal{N} = 4$ gauge theory perturbed by a SO(6) invariant scalar mass and that the high-temperature phase is just the well-known AdS-Schwarzschild solution. We compute, within supergravity, the resulting Hawking-Page phase transition which in this model can be interpreted as a deconfining transition in which the vev for the operator TrF^2 dissolves. In the presence of quarks the model also displays a simultaneous chiral symmetry restoring transition with the Goldstone mode and other quark bound states dissolving into the thermal bath.

INTRODUCTION

The simplest examples of non-supersymmetric deformations of the AdS/CFT Correspondence [1]-[4] are those in which the dilaton has some non-trivial profile in the radial direction of the space [5]-[11]. Since the dilaton carries no R-charge the five-sphere is left intact. The presence in the dilaton profile at large AdS radius, r, of a term of the form $1/r^4$ indicates the presence of a vev for the operator TrF^2 in the $\mathcal{N} = 4$ gauge theory. This operator is the F-term of a chiral superfield so supersymmetry is manifestly broken.

Simple gravitational theories of this type have been shown to generate confining behaviour in Wilson loops in the dual gauge theory and a discrete glueball spectrum [6, 10]. Quarks have also been introduced through D7 brane probes [15]-[17] and chiral symmetry breaking in the pattern of QCD is induced [13, 18]. These models are therefore a nice toy model of a gauge theory that behaves in many respects like QCD.

Given the successes of this model at zero temperature it is interesting to investigate the finite temperature behaviour of the solution. One expects a Hawking-Page type phase transition [19] where, when the temperature grows greater than the perturbation, the vev for TrF^2, a first order transition would occur from the zero temperature solution with a compact time dimension to a black hole geometry. The former has free energy of order one whilst that of the latter is of order N^2 - we would be seeing deconfinement of the gauge degrees of freedom [2, 3]. It would be interesting in addition to understand how chiral symmetry breaking behaves through this transition.

Here we will seek dilaton flow black hole geometries that might describe the high temperature phase of the gauge theory. In fact though we will analytically show that there are no black hole geometries with a non-trivial dilaton in five dimensional supergravity. The only candidate

for the high temperature phase of the dilaton flow geometry, at the supergravity level, is in fact the normal AdS-Schwarzschild geometry. This is of course identical to that describing the high T phase of the $\mathcal{N} = 4$ gauge theory. We conclude that the vev for TrF^2 is an induced operator which dissolves at finite temperature. Taking this as our assumption we compute the Hawking-Page type transition and show that the behaviour matches the usual intuition discussed above, though we must caveat our analysis because we ignore potential string theory corrections to the highly-curved region of the low temperature phase.

We learn that in the low temperature phase the vevs of the glue and quark fields are temperature independent as are the glueball and meson masses. This matches large N field theory arguments made in [20]. The finite temperature solution of the $\mathcal{N} = 4$ theory has also been studied vigorously [26] including in the presence of quarks [17]- all of those results can now be seen to apply to the high temperature phase of the dilaton flow geometry too. In particular the vev of TrF^2 and the chiral symmetry breaking quark condensate (at zero quark mass) both switch off in the high T phase. The Goldstone boson of the symmetry breaking becomes massive and indeed melts into the thermal bath.

The above story though requires some additional explanation. In the low T phase if the vev for TrF^2 is an induced operator what is the perturbation to the $\mathcal{N} = 4$ gauge theory that is breaking supersymmetry? We propose a story that might explain this. Since supersymmetry is broken but $SO(6)_R$ preserved we expect all other $SO(6)_R$ invariant operators to switch on. As was argued in [6] and we will discuss more below, amongst these operators, at large N, is an SO(6) preserving scalar mass term. This mass term is not described by a supergravity field so is essentially invisible in the solutions - there are examples of supersymmetric [21, 22] and non-supersymmetric [24] flows where this operator is present yet invisible in terms of a supergravity field. In those ge-

ometries the mass appears to be generated through the RG flow by a fermion mass term that *is* described by a supergravity mode. Here a simple D3 brane probe shows there is no explicit mass in the dilaton flow geometry though and it is not in this class.

The $SO(6)_R$ invariant scalar mass is described by a string rather than supergravity mode so one naively expects it to describe a super-irrelevant source (if nothing else is there to regenerate it). That is, if present, the source would be invisible in the IR before growing sharply in the UV and dominating the physics. Such a source term would show up as a sharp UV cut off, so the only impact of that cut off in the low energy theory would be the symmetries it imprinted. We propose that the dilaton flow geometry describes the IR physics below such a UV cut off.

When the dual gauge theory is viewed as the $\mathcal{N} = 4$ theory with a scalar mass perturbation it seems a very natural theory to study as a toy for QCD - it is a sensible non-supersymmetric, strongly coupled gauge theory. Of course none of the gaugino super-partners are decoupled from the strong dynamics so it is not QCD.

Our results are also interesting as ten dimensional realizations of the "hard-wall" transitions explored in [20] (the introduction of a hard infra-red wall has long been used as a very simplistic way of introducing a mass gap into the gauge theory [27]). To make that connection stronger we will begin by describing briefly the thermal transition in the $\mathcal{N} = 4$ gauge theory with an $SO(6)$ invariant scalar vev. This should be the ultra-relevant operator that is described by the string mode discussed above. The gravity dual of this theory is precisely $AdS_5 \times S^5$ with a hard IR cut off at some finite radius - the cut off corresponds to the surface of a five-sphere that the D3 branes have been spread evenly over. This case serves as an example of how these scalar operators are invisible in the supergravity solution.

Our main computation here is to show that there is no dilaton flow black hole in five dimensional supergravity and to compute the Hawking-Page transition to a pure AdS-like black hole in that model. Finally we will briefly review the phase structure of the non-supersymmetric gauge theory collating results from elsewhere in the literature.

A HARD WALL - $\mathcal{N} = 4$ SYM ON MODULI SPACE

We begin by creating a true AdS-dual of a hard wall model. The set up is simple - one spreads the D3 branes at the origin ($r = 0$) of the usual AdS/CFT construction onto the surface of a five-sphere centred at the origin.

If a finite number of D3 branes are evenly distributed then the configuration will preserve a discrete sub-group of the $SO(6)$ symmetry group. In the infinite N limit where the distribution becomes smooth the $SO(6)$ group is preserved. In the gauge theory this configuration corresponds to one where

$$Tr\phi_i\phi_j \propto \delta_{ij} \tag{1}$$

which is an $SO(6)$ singlet. The U(N) gauge theory is broken to a $U(1)^N$ theory that is non-interacting (all matter is in the adjoint so uncharged) in the case of a finite number of D3 branes. As the density increases the scalar vevs connecting adjacent sites become small and one has a deconstructed model of a five dimensional U(1) gauge theory living on the five sphere surface - again it is non-interacting.

On the gravity side it is clear from a Gauss' law argument that the geometry does not change (as in the case of a planet being described by the Schwarzschild black hole metric). The dual is just $AdS_5 \times S^5$ with the usual four form (here $L^4 = 4\pi g_s N \alpha'^2$)

$$ds^2 = \frac{u^2}{L^2}dx_4^2 + \frac{L^2}{u^2}du^2 + L^2 d\Omega_5^2 \tag{2}$$

$$C_{(4)} = \frac{u^4}{L^4}dx^0 \wedge ..dx^3 \tag{3}$$

except that the surface of the five sphere of D3 branes acts as a cut off on the space (formally within there is flat space since there are no sources).

The obvious candidate for the finite temperature version of the theory is just AdS-Schwarzschild [3]. Restricting ourselves to a black hole background with temperature equal to $\frac{u_h}{\pi L^2}$ and working with the usual Poincaré coordinates

$$ds^2 = \frac{K(u)}{L^2}d\tau^2 + L^2\frac{du^2}{K(u)} + \frac{u^2}{L^2}dx_3^2 + L^2 d\Omega_5^2 \tag{4}$$

with

$$K(u) = u^2 - \frac{u_h^4}{u^2} \tag{5}$$

We can seek a Hawking-Page transition by comparing the free energy of AdS cut off at the radius of the D3 five sphere, u_0, and with a compact time dimension versus that of the AdS Schwarzschild solution cut off by either the radius of the five sphere or the horizon whichever is largest.

This computation can be performed within the five dimensional truncation of IIB supergravity on this space. The two five dimensional metrics are then simply

$$ds^2 = \frac{u^2}{L^2}d\tau^2 + L^2\frac{du^2}{u^2} + \frac{u^2}{L^2}dx_3^2 \tag{6}$$

$$ds^2 = \frac{K(u)}{L^2}d\tau^2 + L^2\frac{du^2}{K(u)} + \frac{u^2}{L^2}dx_3^2 \qquad (7)$$

with no four forms. The comparison at this level is naively precisely the computation of Herzog [20] which we briefly review. We will see shortly that in the full theory there is an extra contribution to the computation.

The Euclidean action for either cut-off AdS or AdS-Schwarzschild is

$$S = -\frac{1}{4\pi G_5}\int d^5x \sqrt{g}\left(\frac{1}{4}R + \frac{3}{L^2}\right) \qquad (8)$$

On-shell $R = -\frac{20}{L^2}$ for both backgrounds.

One must rescale the time coordinates so as to ensure that the period of the time directions match at the cut off Λ [3]. One then finds the action difference

$$S_{BH}-S_{AdS} = \frac{1}{2G_5 u_h L^3}\left(\int_{u_h}^{\Lambda} u^3 du - \sqrt{\frac{K(\Lambda)}{\Lambda^2}}\int_{u_0}^{\Lambda} u^3 dr\right) \qquad (9)$$

Taking the limit $\Lambda \to \infty$ one obtains $S_{BH} - S_{AdS} = \frac{1}{8G_5 u_h L^3}\left(u_0^4 - \frac{1}{2}u_h^4\right)$. This is Herzog's result.

This computation suggests that if $u_0 > \sqrt[4]{\frac{1}{2}}u_h$ the black hole action is larger and a hard wall solution is favoured - giving Herzog's transition temperature $T_c = \frac{\sqrt[4]{2}u_0}{\pi L^2}$.

In fact the above computation can be interpreted as a valid result in the AdS-QCD approach, in which one constructs phenomenological models without insisting that the holographic physics is an exact realization of IIB SUGRA. To study the actual behaviour of the dual $\mathcal{N} = 4$ gauge theory we must solve the SUGRA problem in its entirety. There is an additional term in the action from the boundary between the flat spacetime within the shell of D3 branes and the AdS geometry outside, which is the difference between the Gibbons-Hawking term from each 'side' of the boundary (we thank Andreas Karch and Steve Paik for pointing this out to us). It is simplest to perform the calculation in the full $10D$ geometry. The Gibbons-Hawking term is $\frac{1}{8\pi G_5}\int_{\Sigma} K d\Sigma$ where the integration is over the boundary Σ and K is the trace of the second fundamental form on the boundary. This is easily evaluated using the relation [25]

$$\int_{\Sigma} K d\Sigma = \frac{\partial}{\partial n}\int_{\Sigma} d\Sigma \qquad (10)$$

The normal derivative is evaluated by setting the metric coefficient of the radial holographic direction to unity (by means of a coordinate transformation) and then differentiating with respect to the radial direction.

The ten-dimensional flat space within the D3 brane shell has line element $ds^2 = L^2\left(d\tau^2 + dx_3^2\right) + \frac{1}{L^2}\left(du^2 + u^2 d\Omega_5^2\right)$ and thus $\int_{\Sigma} d\Sigma = u^5$ up to constants, a multiple of the five-sphere volume and four-space volume. Performing the normal derivative one obtains a contribution of $5u_0^4$.

The $AdS_5 \times S^5$ geometry outside the D3 brane shell has line element $ds^2 = L^2 e^{2r}(d\tau^2 + dx_3^2) + \frac{1}{L^2}\left(dr^2 + d\Omega_5^2\right)$ and thus $\int_{\Sigma} d\Sigma = e^{4r}$ up to volume factors. Performing the normal derivative and transforming back to the 'u'-type coordinates one finds a contribution of $4u_0^4$ times the overall factors.

Including this in our computation we see that it cancels out the total action for the cut-off AdS spacetime entirely leaving

$$S_{BH} - S_{AdS} = \frac{1}{8G_5 u_h L^3}\left(-\frac{1}{2}u_h^4\right) \qquad (11)$$

This implies that the Hawking-Page transition actually takes place at any finite temperature for the theory on its moduli space, which is the same result as for the theory at the superconformal point. Field theoretically this is because the temperature generates a potential for the adjoint scalars of the gauge theory which forces their vev to zero. The transition then naturally occurs immediately above T=0.

DILATON FLOW GEOMETRIES

We now turn to constructing solutions of the supergravity equations of motion with non-trivial dilaton flows.

Five-dimensional action and equations of motion

We will work in $\mathcal{N} = 8$ SUGRA in five dimensions [28]-[30] which is a truncation of IIB string theory on $AdS_5 \times S^5$ and it is known that any solution can be lifted to a complete ten dimensional geometry. The 40 scalars which participate in the superpotential will be set to zero (leaving a constant superpotential which acts as a negative cosmological constant) and we consider a solution with nontrivial dilaton and zero axion.

The effective five-dimensional action is (we use the normalization conventions of [6] and set $L \equiv 1$ for this sec-

tion)

$$S = \frac{1}{4\pi G_5} \int d^5 x \sqrt{-g} \left(\frac{1}{4} R - \frac{1}{8} g^{ab} \nabla_a \phi \nabla_b \phi + 3 \right) \quad (12)$$

The non-extremal background and the background with a nontrivial scalar are both non-supersymmetric and we cannot apply the technique of Killing spinor equations. Instead we use symmetry to constrain the form of the solutions.

We will make an ansatz, following the analysis of the $\mathcal{N} = 2^*$ gauge theory in [31], of the form

$$ds_5^2 = e^{2A} \left(-e^{2B} dt^2 + dx_3^3 \right) + dr^2 \quad (13)$$

The presence of A allows the dilaton to have a non-trivial r dependence and that of B allows for non-zero temperature.

The field equations are

$$\frac{1}{4} R_{ab} = \frac{1}{8} \partial_a \phi \partial_b \phi - g_{ab} \quad (14)$$
$$\nabla^2 \phi = 0 \quad (15)$$

Using the linear combinations $\bar{A} \equiv A + \frac{1}{4}B$ and $\bar{B} \equiv \frac{\sqrt{3}}{4}B$ as in [31] these equations take the form

$$\phi'' + 4\bar{A}'\phi' = 0 \quad (16)$$
$$\bar{B}'' + 4\bar{A}'\bar{B}' = 0 \quad (17)$$
$$6(\bar{A}')^2 = \frac{1}{4}(\phi')^2 + 2(\bar{B}')^2 + 6 \quad (18)$$

The first two equations can be integrated to yield $\phi' = c_1 e^{-4\bar{A}}$ and $\bar{B}' = c_2 e^{-4\bar{A}}$. We will see later that the two solutions we concentrate on correspond to setting either one or the other of these constants zero. Defining $6c_3^2 \equiv (\frac{1}{4}c_1^2 + 2c_2^2)$ we obtain

$$(\bar{A}')^2 = c_3^2 e^{-8\bar{A}} + 1 \quad (19)$$

These equations are analytically solvable with solutions

$$e^{4\bar{A}} = \frac{c_4^2 e^{8r} - c_3^2}{2 c_4 e^{4r}} \quad (20)$$

$$\bar{B} = \frac{c_2}{4c_3} \ln \left(\frac{c_4 e^{4r} - c_3}{c_4 e^{4r} + c_3} \right) + B_0 \quad (21)$$

$$\phi = \frac{c_1}{4c_3} \ln \left(\frac{c_4 e^{4r} - c_3}{c_4 e^{4r} + c_3} \right) + \phi_0 \quad (22)$$

For any solution that returns to AdS asymptotically $B_0 = 0$ and ϕ_0 is the dilaton value in the AdS limit.

Solution with no event horizon

Let us first take the solution above and set $B \equiv 0$ to find a zero temperature dilaton flow with manifest 4D Lorentz invariance. The solution can be recast (by setting $c_3 = c_4\zeta$) in the form

$$e^{2A} = \sqrt{\frac{c_4}{2}} \sqrt{e^{4r} - \zeta^2 e^{-4r}} \quad (23)$$
$$\phi = \sqrt{\frac{3}{2}} \ln \left(\frac{e^{4r} - \zeta}{e^{4r} + \zeta} \right) + \phi_0 \quad (24)$$

To match to other results in the literature [13] one can rescale the x_4 coordinates to effectively set $c_4 = 1/2$, set $2u^2 = e^{2r}$ and $\zeta = -4u_0^4$. Reinstating the five sphere and moving to string frame one arrives at the 10D metric

$$ds^2 = e^{\phi/2} \left(\frac{u^2}{L^2} \mathcal{A}^2(u) dx_4^2 + \frac{L^2}{u^2} du^2 + L^2 d\Omega_5^2 \right) \quad (25)$$

with

$$\mathcal{A}(u) = \left(1 - \left(\frac{u_0}{u} \right)^8 \right)^{\frac{1}{4}}, \quad e^\phi = \left(\frac{(u/u_0)^4 + 1}{(u/u_0)^4 - 1} \right)^{\sqrt{3/2}} \quad (26)$$

The four form is just that in pure AdS. This metric clearly becomes $AdS_5 \times S^5$ at large u and has a deformation parameter u_0^4 which has dimension four and no R-charge - this parameter is naturally identified with TrF^2 in the gauge theory. Since TrF^2 is the F-term of a chiral superfield supersymmetry is therefore broken in this gauge theory.

A crucial aspect of the geometry is that it is singular at $u = u_0$ with the dilaton blowing up. A singularity should be a source of unease and we do not have a full explanation of it but we wish to argue there are number of ideas that suggest such geometries are worthy of study none the less. The presence of D3 branes in the geometry do provide sources that, in some non-supersymmetric configuration, might complete the geometry (compare to the hard wall model where they account for the discontinuity between AdS and flat space). The N=2* geometry [22]is also singular at a point where the effective gauge coupling diverges - this geometry has been matched to the expected field theory solution at a particular point on moduli space [23]. This model provides evidence that a divergent gauge coupling can show up as a divergence in the geometry.

Our real motivation for the continued study here though is the phenomenological successes of the geometry. It has been shown to be confining in [6, 10] and to break chiral symmetries when quarks are introduced [13] as we will discuss below. In this sense we can think of it as a back

reacted hard wall with the correct properties to describe QCD-like physics.

Black hole geometry

Let us now turn to finding the high temperature phase of the dilaton flow geometry just explored.

To have a solution with a horizon we will choose constants such that the function B goes as a constant plus $\ln r$ near $r = 0$ ($\bar{B} \sim \sqrt{3}/4 \ln r$). From (21) this gives the constraints

$$c_2 = \sqrt{3}c_3, \qquad c_4 = c_3 \qquad (27)$$

At this point we note that, with the definition of c_3 ($6c_3^2 \equiv (\frac{1}{4}c_1^2 + 2c_2^2)$), c_1 vanishes so the dilaton profile in the non-extremal solution is just a constant. This result is a simple example of a 'no hair' theorem.

We have learnt that there is no black hole solution with a radially dependent dilaton. This means there is not a gravity dual of a high temperature theory with TrF^2 switched on. One's immediate response is to become worried that if the $\mathcal{N} = 4$ gauge theory cannot be perturbed by a vev for TrF^2 at finite temperature then the zero temperature model is suspect. In fact though we believe it is telling us that the vev for TrF^2 is not the fundamental perturbation but an operator induced by the dynamics. We will discuss what the true perturbation might be in the next section.

We are left with a unique black hole solution which in the unbarred quantities is thus

$$e^{2A} = \frac{c_4}{2}\left(e^{2r} + e^{-2r}\right) \qquad (28)$$

$$B = \ln\left(\frac{e^{4r} - 1}{e^{4r} + 1}\right) \qquad (29)$$

$$\phi = \phi_0 \qquad (30)$$

This geometry should describe the high temperature phase of the dilaton flow theory. In fact by rescaling x_4 to set $c_4 = 2u_h^2$ and defining $e^{2r} = \frac{u^2 + \sqrt{u^4 - u_h^4}}{u_h^2}$ (which are asymptotically the same choices as for the dilaton flow geometry) this solution can be reduced to the usual Poincaré coordinate form of five-dimensional AdS-Schwarzschild (4) with Hawking temperature $T_H = \frac{u_h}{\pi}$. We conclude that the scalar mass deformed gauge theory shares the same supergravity description at finite temperature as the unperturbed $\mathcal{N} = 4$ gauge theory! A similar conclusion was reached in [14] in the context of adding a dilaton to the hard wall model.

THE ORIGIN OF SUPERSYMMETRY BREAKING

We now turn to the question of the origin of supersymmetry breaking in the dilaton flow geometry if TrF^2 is an induced operator as it appears it must be from the above analysis.

Since supersymmetry is broken, yet $SO(6)_R$ preserved, in the T=0 geometry, we expect all $SO(6)$ invariant operators to be present. Amongst these $SO(6)$ invariant operators is an equal mass for each of the six scalar fields - one would expect the scalar masses to rise to the scale of the supersymmetry breaking scale. Such an $SO(6)$ invariant mass is invisible in the supergravity solution for the same reasons as the $SO(6)$ scalar vev operator discussed above. A frequently argued interpretation of the fact that this source is not described by a supergravity mode is that it is a super-irrelevant operator. The vev for the scalar operator discussed in the hard wall model above would then be super-relevant; that is it would have no impact on the UV of the theory until suddenly at some point in the IR it would dominate it - this can certainly be matched to it's appearance as a sharp IR cut off on the geometry. In this language one would expect the mass term to show up as a sharp cut off on the space at some large radius or UV scale. Below that scale it would naively have no impact on the dynamics. This is not quite true though because it would define the symmetries of the theory below that UV cut off and in particular leave a non-supersymmetric but $SO(6)$ invariant flow at lower energies. The dilaton flow geometry is the natural candidate for this flow. In this interpretation one should cut off the dilaton flow at some point in the UV, although this point could presumably be set at an arbitrarily high scale. In analogy with the hard wall model the point where the cut off appears would be undetectable in the low energy flow.

The above seems a consistent interpretation but the supersymmetric N=1* [21] and N=2* [22] theories suggest a more complicated story is also possible. In those theories precisely the scalar mass discussed here is present in the Lagrangian of the gauge theory yet no supergravity mode directly represents it in the supergravity duals. The scalar mass must, by supersymmetry, be present and tied to the fermion masses (in superspace there is only the one mass parameter) which are described by supergravity fields. The flows for these fields show the mass term to be relevant. These are therefore examples of theories with a relevant scalar mass present but no explicit dual operator to indicate it. The Yang Mills* geometry [24] is a non-supersymmetric example - a fermion mass term is introduced there yet the potential for a D3 probe shows there to be a scalar mass present too. This is the simple way for us to test whether there is a scalar mass present

here - we look at the potential for moving a D3 probe in the transverse directions of the dilaton flow geometry. The result is clear on simple dimensional grounds - the asymptotic potential for the D3 motion must go like u_0^4 (note the fourth power is the lowest to occur in the metric) to some positive power so that it vanishes when u_0 does. We can only have a u dependent term of the form $V \sim u_0^8/u^4$ - there is no $m^2 u^2$ term and so no explicit mass. The full D3 potential is given by

$$V \sim u^4 \mathcal{A}^4 - u^4 \sim -\frac{u_0^8}{u^4} \qquad (31)$$

which is stable.

Presumably in the theories with fermion masses the scalar mass is continually regenerated through the RG flow whilst if only the scalar mass is present it flows to zero in the IR. We conclude that in the dilaton flow geometry the scalar mass would indeed only be visible through a sharp UV cut off as discussed above. This seems a consistent interpretation to us and with this in mind we will go on to analyze the high temperature transition in the dilaton flow geometry.

Note this geometry is also closely related to those of Constable and Myers [11] which have in addition non-trivial u dependence in the four form - the existence of this larger class of geometries suggest that there are multiple SO(6) invariant string modes that are invisible in the supergravity and that determine the dynamics. In the field theory one can imagine higher dimension operators and so forth that could play a role. These geometries typically also show confinement and chiral symmetry breaking though [18].

THERMODYNAMIC COMPUTATION

One way to test our assertion that AdS-Schwarzschild is the high temperature phase of the non-supersymmetric deformation of the $\mathcal{N} = 4$ gauge theory is to check the Hawking-Page phase transition makes sense. We will compute the Euclidean action for both solutions, specifying a black hole horizon at $u = u_h$ and a dilaton flow singularity at $u = u_0$.

To make the comparison fair we must set the parameter c_4 equal in the two geometries so they have the same large-r AdS limit. We will perform the calculation in the Schwarzschild-type coordinates, rescaling the Euclidean time coordinate for the dilaton flow geometry as for our hard-wall calculation. Both geometries asymptote to AdS_5 so we can set the same UV cut-off Λ in both cases, before taking the limit $\Lambda \to \infty$.

Our interpretation above that the dilaton flow geometry is the IR theory below some UV cut off associated with the presence of a scalar mass means that formally we should keep the UV cut off fixed. We can though imagine that that scale is arbitrarily high. In any case we will give the result for arbitrary Λ below.

The Euclidean action density per unit spatial volume for the black hole solution is

$$S_{BH} = -\frac{1}{4\pi G_5} \int_0^{\frac{\pi L^2}{u_h}} d\tau \int_{u_h}^{\Lambda} \sqrt{-g} \left(\frac{1}{4}R + \frac{3}{L^2}\right) dr \quad (32)$$

The trace of the Einstein equation gives $R = -\frac{20}{L^2}$ so

$$S_{BH} = \frac{1}{2G_5 u_h L^3} \int_{u_h}^{\Lambda} u^3 du = \frac{1}{8G_5 u_h L^3} \left(\Lambda^4 - u_h^4\right) \qquad (33)$$

The Euclidean action density per unit spatial volume for the dilaton flow solution is, having used the trace of the equation of motion to remove the scalar gradient term and allowing for the rescaling of Euclidean time, simply

$$S_{DF} = \frac{1}{2G_5 L^2} \int_0^{\frac{\pi L^2}{u_h}\sqrt{1-\frac{u_h^4}{\Lambda^4}}} d\tau \int_{u_0}^{\Lambda} \sqrt{-g} \, dr \qquad (34)$$

This is

$$S_{DF} = \frac{1}{2G_5 u_h L^3} \sqrt{1 - \frac{u_h^4}{\Lambda^4}} \int_{u_0}^{\Lambda} \left(u^3 - \frac{u_0^8}{u^5}\right) dr$$

$$= \frac{1}{8G_5 u_h L^3} \sqrt{1 - \frac{u_h^4}{\Lambda^4}} \left(\Lambda^4 + \frac{u_0^8}{\Lambda^4} - 2u_0^4\right) \quad (35)$$

Hence, in the $\Lambda \to \infty$ limit, the difference in the actions is simply

$$S_{BH} - S_{DF} = \frac{1}{16G_5 u_h L^3} \left(4u_0^4 - u_h^4\right) \qquad (36)$$

For a deformation scale $u_0 > \frac{u_h}{\sqrt{2}}$ the dilaton flow solution is thermodynamically favoured whereas for a deformation scale $u_0 < \frac{u_h}{\sqrt{2}}$ the AdS-Schwarzschild solution is favoured. The transition temperature is $T_c = \frac{\sqrt{2}u_0}{\pi L^2}$.

The phase transition appears to make complete sense with our interpretation of the high and low temperature phases. For temperatures below the value of the supersymmetry breaking scale u_0 (up to a factor of order unity) the non-supersymmetric SO(6) invariant scalar mass deformed $\mathcal{N} = 4$ gauge theory is described by the dilaton flow geometry with an induced vev for TrF^2. As the temperature passes through the supersymmetry breaking scale there is a transition to the deconfined plasma

described by AdS-Schwarzschild - here the vev of TrF^2 is zero.

It is important to stress though that the computation above may not be complete. We saw above that when the hard wall model was converted into a full string geometry a Gibbons-Hawking term appeared at the IR singularity that played a crucial role in the thermodynamics of the $\mathcal{N} = 4$ theory on moduli space. One must worry that a similar surface term might appear if the dilaton flow geometry were completed to a fuller string theoretic understanding. In addition the negative term in our action computation is dominated near the singularity and might also change were the singularity resolved in someway. Within supergravity, our only available tool, it is hard to see how to address these complications - it is encouraging though that the calculation as is does agree with expected field theory intuition and the naive hard wall geometry as applied to QCD. It is interesting to compare this case to the AdS-QCD models of [12] in which the action computation is dominated away from the IR singularity - those theories may have better control.

Another interesting point is that the supergravity computation naively suggests the dilaton flow geometry's action is lower than that of AdS! Our discussion of the origin of supersymmetry breaking though suggests this is not a correct comparison - we have argued that the geometry only applies below some UV cut off corresponding to the scale where a super-relevant scalar mass becomes important. It is an artefact of supergravity that this cut off is invisible in the IR. Above that cut off the presence of a non-normalizable mode would make the dilaton flow geometry's action much greater than the case of AdS extended to infinity. We do not expect the $\mathcal{N} = 4$ theory to spontaneously break supersymmetry. This does not affect our computation since both the AdS Schwarzschild black hole and the dilaton flow geometry share the same UV (at least if the cut off is far enough into the UV) and hence would see the same UV cut off physics.

ASPECTS OF THE PHASE TRANSITION

The identification of AdS-Schwarzschild as the high T phase of a non-supersymmetric theory and the dilaton flow as the low temperature phase of that same theory is quite remarkable. Our findings go some way towards explaining why the AdS-Schwarzschild gravity dual is a reasonable toy model of high-temperature QCD whereas the zero-temperature supersymmetric D3-D7 model is quite unlike low-temperature QCD (it is conformal, for example, when the quarks are massless). Both of the geometries have been studied in detail already in the literature, including in the presence of quarks, and we can look at a number of properties of the transition.

Below the critical temperature, the core of the dilaton flow geometry is repulsive to strings. The result is that a Wilson line calculation shows there to be a linear quark-antiquark potential as the string falls towards $r = 0$ before settling a little away from the singularity at u_0 [6, 13]. In keeping with the implied confinement there is a discrete spectrum of glueballs which are the eigenmodes of the Klein-Gordon equation on the geometry for the usual plane-wave ($\propto e^{ik \cdot x_4}$) ansätze. Table I shows the masses of the lowest five scalar glueball states, in units of u_0/L^2 (we can reproduce the field equations for these fluctuations in [6] but disagree on the numerical values for the masses) [48]

n	1	2	3	4	5
M_n	4.1	7.2	10.2	13.2	16.2

Table I: Lowest five glueball masses in the zero temperature dilaton flow geometry in units of the deformation scale u_0/L^2.

n	ω_n
1	$\pm$ 3.119452 - 2.746676 i
2	$\pm$ 5.169521 - 4.763570 i
3	$\pm$ 7.187931 - 6.769565 i
4	$\pm$ 9.197199 - 8.772481 i
5	$\pm$ 11.202676 - 10.774162 i

Table II: Lowest five glueball quasi-normal modes in the AdS-Schwarzschild geometry in units of $\frac{u_h}{L^2}$.

Above the critical temperature, the theory is in a deconfined phase. There is no longer a spectrum of glueball normal modes, rather the gravity dual admits a spectrum of unstable quasinormal modes which was calculated in [47]. The field theory interpretation of the quasinormal spectrum is to give the mass and decay width for a glueball excitation embedded in a thermal bath of SYM plasma. The finite decay timescale can be viewed as the timescale for the 'melting' of the glueball state. The breaking of Lorentz symmetry means there is a nontrivial dispersion relation $\omega(k)$ for a scalar glueball excitation [47]. Table II shows the lowest five quasinormal frequencies, which are measured in units of $\frac{u_h}{L^2}$ which is $\propto T$ - the natural scale of the lowest quasinormal frequency is the temperature.

Flavour degrees of freedom can be included by embedding D7 branes in each of the geometries discussed [17, 35, 36] - the results to date in these geometries use the probe or quenched limit [15]. The asymptotic profile of the D7 encodes the relationship between the hard quark mass m_q and the expectation value of the quark condensate $\langle \bar{q}q \rangle$

[18]. Mesonic modes are dual to fluctuations derived from the DBI action of the D7 [16].

Below the critical temperature we find the probe D7 always wants to lie outside the deformation scale of the dilaton flow geometry for the embeddings of physical interest (usefully avoiding the singular region of the geometry). One finds there is a nonzero value of $\langle \bar{q}q \rangle = 1.51 u_0^3$ for zero m_q, indicating spontaneous breaking of a $U(1)_R$ symmetry of the model (this symmetry is analogous to the axial $U(1)_A$ of QCD) - this is a nice model of QCD-like behaviour since the dynamics of the quark condensate generation is included even if the full non-abelian chiral symmetry breaking is not present. There are discrete spectra corresponding to pion-like and sigma-like scalar excitations [10]. The lowest pion-like state is massless for $m_q = 0$ and its mass grows in accordance with the Gell-Mann-Oakes-Renner relation for small m_q. In addition there is a tower of massive vector meson excitations dual to the Maxwell field on the D7 worldvolume. The numerical values for all these meson masses tend to the no-deformation result (equation (3.19) in [16]) for large m_q, that is $M \sim 2\sqrt{2}m_q/L^2$ - we list them in Table III and display them in Figure I in units of u_0.

m_q	$M_\pi L^2$	$M_\sigma L^2$	$M_{vector}L^2$
0.10	0.7	3.1	2.9
0.50	1.9	3.5	3.3
1.00	3.1	4.1	3.9
2.00	5.7	6.0	6.0
3.00	8.5	8.6	8.6
4.00	11.3	11.4	11.4

Table III: the mass of the pion, sigma and rho meson modes as a function of the quark mass in the low temperature dilaton flow geometry - all in units of u_0.
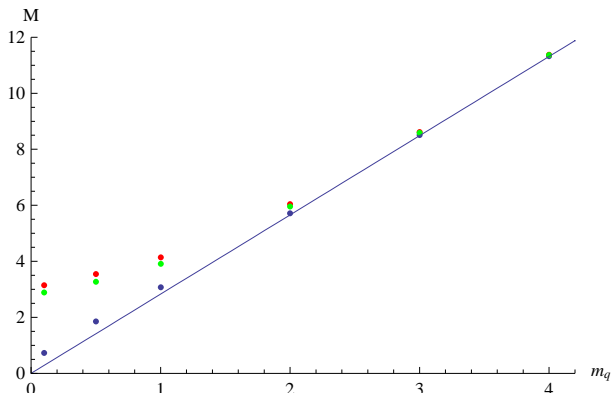

FIG. 1: Pion (blue), sigma (red) and vector (green) masses as a function of quark mass - all in units of u_0. The line shows the large-m_q limit.

Above the critical temperature the physically relevant D7

embeddings in the black hole geometry [18, 37, 38, 39, 40, 41] give $\langle \bar{q}q \rangle = 0$ for $m_q = 0$ - there is no chiral symmetry breaking and hence no pion-like meson. The D7 can either end on the black hole horizon (small m_q) or for large enough m_q it has sufficient tension to support itself away from the black hole - there is a first-order phase transition in the behaviour of (quenched) quark matter as one raises the quark mass in the plasma background. In the former case there are quasinormal spectra representing the melting of scalar and vector mesonic excitations in the hot background [33, 43, 44]. In the latter case there are discrete spectra of scalar and vector meson masses with scale set by m_q which tend to the values in [16] as $m_q \gg T$. Our transition behaves in the same way as long as the quarks are sufficiently light ($m_q < 0.92\frac{\sqrt{\lambda}T}{2}$) [33]. In the undeformed theory this bound implies that the mesons melt once the temperature of the background becomes of order the meson mass since the meson masses are $\sim \frac{m_q}{\sqrt{\lambda}}$. In our case the pion-like meson is an exception to this - one can have a massless pion that does not 'melt' until the background reaches some finite temperature.

n	ω_n
1	$\pm$ 2.1988 - 1.7595 i
2	$\pm$ 4.2119 - 3.7749 i
3	$\pm$ 6.2155 - 5.7773 i
4	$\pm$ 8.2172 - 7.7781 i
5	$\pm$ 10.2181- 9.7785 i

Table IV: the scalar mesonic quasinormal frequencies in the high T phase ($m_q = 0$) - in units of $\frac{u_h}{L^2}$.

There are recent results concerning a lower bound for the ratio of viscosity to entropy density of a strongly-coupled field theory [34], $\frac{\eta}{s} \geq \frac{1}{4\pi}$, where the equality is for the deconfined phase of $\mathcal{N} = 4$ SYM theory (for a review see [26]). Our findings show that strict equality also applies to certain non-supersymmetric theories in their deconfined phase, physically due to the universality of this phase in the large-N limit.

We can perform an estimate of the deconfinement temperature in our model. The mass of the lowest-lying vector state for zero quark mass can be compared to the mass of the ρ meson, experimentally 776MeV. The vector mass is $2.80 u_0/L^2$ and the deconfinement temperature is $T_c = \frac{\sqrt{2}u_0}{\pi L^2}$. This gives an estimate for the deconfinement temperature of $T_c \sim 124$ MeV. This is very similar to the estimate produced by the 'hard-wall' model [20] and is somewhat low compared to real QCD.

CONCLUSION

We have found the finite-temperature version of a chiral symmetry breaking dilaton flow. Performing a calculation at the level of classical supergravity we found that for a temperature sufficiently high that the black hole radius is greater than the deformation scale, the geometry undergoes a first-order phase transition to the AdS-Schwarzschild geometry. From the gauge-theory perspective, we have argued that the dilaton flow solution results from turning on a $SO(6)$ invariant scalar mass deformation. This deformation is not described by a SUGRA mode and is probably present only as a UV cut off that determines the symmetries of the IR theory, but one does observe that the non-trivial dilaton profile describes a running coupling and the operator TrF^2 is 'induced'. We have found that at high temperature this does not happen and TrF^2 remains zero. Incorporating other results already in the literature one can see that the transition corresponds not just to a deconfinement transition but also a simultaneous chiral symmetry restoration transition if (quenched) quarks are introduced into the theory. We believe this is the simplest, four dimensional, AdS/CFT derived caricature of a QCD-like theory.

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[48] The dilaton equation of motion is given by $\partial_u(u^5\mathcal{A}^4\partial_u\phi) + u\mathcal{A}^2 M^2\phi = 0$ (here we write u in units of u_0 and rescale x_4 so that factors of L, u_0 are common to the metric). The UV solutions take the form $\phi \sim c_1 + c_2/u^4$ with the latter being required for a glueball fluctuation. In the IR the equation can be recast in Schrödinger form - we write $u = 1 + z$, then

change coordinates to y such that $dy/dz = 1/(8z)^{1/4}$, and finally write $\phi = uv$ with $\frac{1}{v}\frac{dv}{dy} = -3/(8z)^{3/4}$. The potential is then of the form $V = -\frac{1}{4y^2}$. Such a potential is of the limiting form that possesses a discrete spectrum bounded from below (see [45] or chapter 5 of [46]). The IR solutions, written in the original u coordinates are of the form $\phi \sim c_3 + c_4\ln(u-1)$ - the former are the physical solutions, the latter blow up and are therefore inconsistent with linearization. All of this is in complete agreement with the analytic discussion in [6] and we can numerically shoot from both the IR and UV solutions to find the values of M for which solutions match to the required UV and IR boundary conditions. We disagree with [6] on the numerical values of these solutions though.